# Section 5.4: The Fundamental Theorem of Calculus

Math 1552 lecture slides adapted from the course materials
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→ Complete the TurningPoint poll on Canvas (NOT graded)

# Today’s Learning Goals

- Know the statements of the FTC and the Second FTC
- Apply the FTC to evaluating definite integrals using the formulas from Section 4.8
- Apply the Second FTC to differentiate an integral

# Theorem: The $2^{nd}$ FTC

Let $f$ be a continuous function on the interval $[a,b]$.

Then if

$$F(x) = \int_a^x f(t)\,dt,$$

$F'(x)=f(x)$ for all $x$ in $(a,b)$.

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\underline{\int_a^x f(t)\,dt}\right] = f(x).}$$

(important to know)

# Example 1: Find F'(2).

$$F(x)=\int_1^x \frac{t}{t^3+3}\,dt$$

A. 2/7

B. $\boxed{2/11}$

C. ¼

D. 3/44

$$f(t) = \frac{t}{t^3+3}, \text{ by FTC}$$

$$F'(x) = f(x)$$

$$\Rightarrow F'(2) = f(2) = \frac{2}{8+3} = \frac{2}{11}$$

# Antiderivatives

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From the second FTC, if

$$F(x)=\int_a^x f(t)\,dt,$$

then $F$ is an antiderivative of $f$.

# The FTC

$F' = f \quad , \quad G' = g$

***The Fundamental Theorem of Calculus***:

Let *f* be a function that is continuous on the interval [*a*,*b*], and let *F* be any antiderivative of *f*. Then:

$$\int_a^b f(x)dx = \underbrace{F(x)\Big|_a^b = F(b) - F(a)}.$$ $(*)$

# Example 2: Evaluate.

$$\int_1^3 \frac{1}{x^2}\,dx$$

A. $\boxed{2/3}$

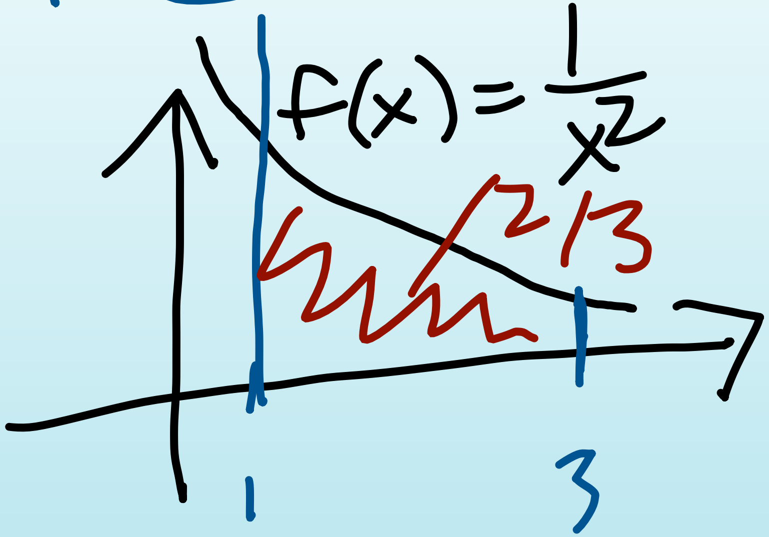
B. 4/3

C. 26/9

D. 26/81

$$\int \frac{dx}{x^2} = \int x^{-2}\,dx = -\frac{1}{x} + C$$

$$F(x) = -\frac{1}{x} + C$$




By the FTC$_1$

$$\int_1^3 \frac{dx}{x^2} = F(3) - F(1)$$

$$= -\frac{1}{3} - \left(-\frac{1}{1}\right) = 2/3$$

# Example 3:

The percent of toxin in a lake, where time is in years, is given by the function:

$$f(t) = 50\left(\frac{1}{4}\right)^t.$$

Find the average amount of toxin in the lake between years 1 and 3.

What are we being asked to find?

$$AV = \frac{1}{3-1}\int_1^3 f(t)\,dt$$

→ eval. the indef. integral first:

$$50\int 4^{-t}\,dt = 50\int e^{-t\cdot\log 4}\,dt$$

$$= \frac{-50}{\log 4}4^{-t} + C$$

$\rightarrow$ use the FTC with

$$F(x) = \frac{-50}{\log 4} 4^{-x} + C$$

$$\Rightarrow AV = \frac{1}{2}\left(F(3) - F(1)\right)$$

$$= \frac{-25}{\log 4}\left(\frac{1}{64} - \frac{1}{4}\right)$$

$$= \frac{-25}{\log 4}\left(\frac{1}{64} - \frac{16}{64}\right)$$

$$= \frac{25 \cdot 15}{64 \cdot \log(4)}$$

# Example 4: Extension to 2nd FTC (chain rule)

Use this extension :

*(Does every one see where this comes from?)* $(*)$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right] = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

to find $F'(x)$ if $F(x) = \int_{3x}^{\cos x} \frac{1}{1+t}dt.$

# Where the extension comes from?

$$F(x) = \int_c^x f(t)\,dt \longrightarrow \text{by the FTC,}$$

(second version)

$$\int_{a(x)}^{b(x)} f(t)\,dt = F(b(x)) - F(a(x))$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = F'(b(x))\cdot b'(x) - F'(a(x))\cdot a'(x) \quad (*)$$

Q]: find $F'(x)$ if $F(x) = \int_{3x}^{\cos x} \underbrace{\frac{dt}{1+t}}$

$f(t) = \frac{1}{1+t}$

$\rightarrow$ apply (*):

$a(x) = 3x, \quad a'(x) = 3$

$b(x) = \cos(x), \quad b'(x) = -\sin x$

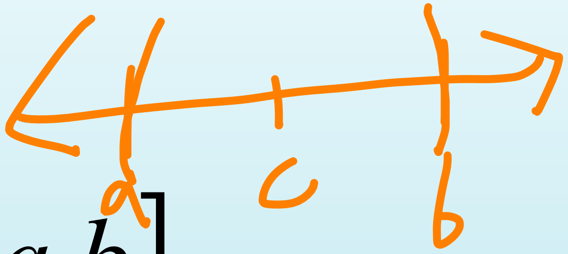
recall by FTC: $F'(x) = f(x)$

$F'(x) = f(\cos x)(-\sin x) - f(3x) \cdot 3$

$$= -\frac{\sin x}{1+\cos x} - \frac{3}{1+3x}$$

# Mean Value Theorem

*MVT for Integration (statement)*:




Let $f$ be continuous on $[a,b]$.

Then there exists a $c \in (a,b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

$$\Longleftrightarrow f(c) = \overbrace{\frac{1}{b-a}\int_a^b f(x)dx}^{\text{avg. value (AV)}}$$

# Example 5:

1 Find the average value of the function:

$$f(x) = 1 - x^2, \ -1 \leq x \leq 3.$$

ON $[a, b]$ $= [-1, 3]$

2 Then find a *c* that satisfies the MVT for integration.

① $AV = \frac{1}{3-(-1)} \int_{-1}^{3} (1-x^2)\,dx$

$$= \frac{1}{4} \underbrace{\left( x - \frac{x^3}{3} \right)}_{F(x)} \Bigg|_{-1}^{3}$$

(by FTC) $$= \frac{1}{4} \left( \underbrace{\left(3 - \frac{27}{3}\right)}_{F(3)} - \underbrace{\left(-1 + \frac{1}{3}\right)}_{F(-1)} \right)$$

$$= \frac{1}{4} \left( -6 + 2/3 \right) = -4/3$$

(2) we are asked to find: $-1 < c < 3$ so that

$$f'(c) = 1 - c^2 = AV = -4/3$$

$$\rightarrow 1 - c^2 = -4/3$$

$$\leftrightarrow 1 + 4/3 = c^2$$

$$\leftrightarrow 7/3 = c^2$$

$\longleftrightarrow \quad c = \pm\sqrt{\frac{7}{3}}$ , exclude

$-\sqrt{7/3}$ since

it is outside

$(-1, 3)$

$\longrightarrow \boxed{c = \sqrt{\frac{7}{3}}}$